The Associative Property

(Also available in WeScheme)

Students explore Associative Property, using Circles of Evaluation as a tool to visualize groupings.

Lesson Goals	Students will be able to… • Recognize that, no matter how we group numbers being either added or multiplied, the end result will always be the same. • Use the Associative Property to simplify expressions and make computation easier.
Student-facing Lesson Goals	• Let's explore the Associative Property via Circles of Evaluation.
Prerequisites	• Simple Data Types • Translating Between Words and Math • Contracts • Equivalence • The Commutative Property
Materials	• PDF of all Handouts and Page • Commutativity and Associativity Starter File • Printable Lesson Plan (a PDF of this web page)
Supplemental Materials	• Additional Printable Pages for Scaffolding and Practice

Key Points For The Facilitator	• The Associative Property - like other laws of arithmetic - is more meaningful and memorable when students view it as a powerful tool to help them to do math efficiently. • Students may need support in parsing expressions that include both an operation that *does* support Associativity alongside an operation where Associativity does *not* apply.

Glossary

Associative Property :: When adding three numbers or multiplying three numbers, it does not matter whether you start with the first pair or the last. The same is true when either adding or multiplying four numbers, five numbers, etc.

Commutative Property :: For any expression involving only addition or only multiplication, changing the order of the numbers will not change the result.

equivalent :: expressions are equivalent when they simplify to the same value, no matter what value is assigned to their variables (if there are any)

The Associative Property *30 minutes*

Overview

Students explore Circles of Evaluation to discover that no matter how we group numbers being either added or multiplied, the end result will always be the same.

Launch

We've already learned about the ***Commutative Property***, which tells us that, for any expression involving only addition or only multiplication, changing the order of the numbers will not change the end result. But what about changing the *groupings*?

- Turn to Discover the Associative Property (1). Look at the first problem.

- What do you Notice about the three Circles of Evaluation? How about the two expressions that accompany them? What do you Wonder?

- Work through the remaining problems with your partner. Be sure to evaluate the Circles to see if they produce equivalent outcomes.
- When you're finished, move on to Discover the Associative Property (2).

- What did you observe about the Commutative Property?

Confirm that students are comfortable with the ***Associative Property***. Specifically, they should recognize that, when adding three numbers, they can start by adding the first pair of numbers *or* the last pair of numbers. Similarly, when multiplying three numbers, students can start by multiplying the first pair of numbers *or* the last pair of numbers. Importantly:

The Associative Property holds for addition and multiplication, but not division or subtraction.

Students apply this understanding in the subsequent activities.

Investigate

- On Associative Property Table, apply the ***Associative Property*** to create ***equivalent*** Circles of Evaluation. Pay attention to the operators!

- Now try True or False? Associative Property. Determine if each equation is true, and be sure to explain your response.

Note: In these activities, students will need to broaden their definition of the Associative Property, as they will encounter expressions with more than three numeric values. The same general rules still apply: When we change the groupings of numbers being added or multiplied, we get the same outcome.

Follow up with students, paying specific attention to expressions with multiple operations. In these instances, students sometimes *can* apply the Associative Property - but only on the parts of the expression that feature only addition or only multiplication.

Synthesize

- Can you ever apply the Associative Property on an expression that includes division or subtraction? Explain.
- When applying the Associative Property, did you ever change the order of the numbers? Why or why not?
- When applying the Associative Property, did you ever change the architecture of the circles? (For instance, did you ever change "three nested circles" to "two-circles-in-a-circle"?) When was that necessary? Why?

The Associative Property and Mental Math *25 minutes*

Overview

Students explore how the Associative Property and Circles of Evaluation can make mental computation simpler.

Launch

- Turn to So Many Groupings!.
- First, study the example. Notice how the groupings change but the order stays the same. Can you think of a fourth way to re-group the expression?
- Now it's your turn! You will explore different ways to group this expression: $98 + 3 + 7 + 26 + 4$
- Draw as many equivalent circles of evaluation as you can in the boxes below, then answer the questions at the bottom of the page. You may change the groupings, but not the order!

Allow students to work, encouraging them to try different groupings and *structures*, all of which will produce the same outcome! If possible, invite students to draw their Circles of Evaluation on the board so that the class can see all possible configurations. After students have drawn their Circles of Evaluation and had some time to consider the reflection questions, follow up with a class discussion.

- Which Circle of Evaluation seemed like it would be the most difficult to solve in your head? Why?
- Which Circle of Evaluation seemed like it would be the easiest to solve in your head? Why?

Investigate

- Complete Which Grouping Makes the Computation Easier?. Examine each Circle of Evaluation and put a check mark by the one that results in a simpler computation. Then, evaluate the expression.
- Now, try Associativity Makes Computation Easier (1) to apply the Associative Property to expressions with more numbers and varying operators.

Synthesize

- How can the Associative Property help you do mental math more efficiently?
- Did Circles help you to visualize associativity? Why or why not?

Applying the Commutative and Associative Properties

Launch

When asked to multiply $2 \times 17 \times 5$, a student who notices that $2 \times 5 = 10$ will arrive at a result much more efficiently than a student who does not.

Rewriting $2 \times 17 \times 5$ as $2 \times 5 \times 17$ is an example of applying *two* properties in one go: reordering the numbers makes way for more helpful groupings. It's easier to multiply 10 by $\{17\}$ than it is to multiply 34 by 5.

The freedom to solve in a variety of ways rather than just moving left to right opens up a world of possibility. In the short term, we can compute efficiently. In the long term, confidence using laws of arithmetic provides a strong foundation for more complex algebraic reasoning.

Investigate

- Turn to Restructuring Addition Expressions, where you will reorder and regroup a given addition expression using a Circle of Evaluation.
- Now, complete Restructuring Multiplication Expressions, where you will reorder and regroup a multiplication expression using a Circle of Evaluation.
- Ready for a challenge? *Optional:* Try Associativity Makes Computation Easier (2) to apply the evaluate expressions with fractions and decimals.

- What was your strategy for restructuring the expressions to make them simpler to evaluate?

With some familiarity of regrouping and reordering, students are ready to get creative and develop their own arithmetic expressions.

- Think of an addition or multiplication problem that *appears* to be very challenging, but is *much* easier to solve after applying the Associative Property *and* the Commutative Property. Write it down on a piece of paper. Be creative!
- Trade papers with a partner. How do your problems compare?
- Represent your partner's expression with a Circle of Evaluation that makes solving simpler.
- Turn your paper in to your teacher.

We encourage you to review students' submissions, and write a few on the board to discuss as a class. We want to sharpen students' eyes and help them develop the ability to spot instances when they might apply the Associative Property in any context.

Synthesize

- How would you describe the relationship between the Associative Property and the Commutative Property? Do you think one is more powerful than the other?
- How are the Commutative and Associative Properties similar? How are they different?

Programming Exploration: Associativity *20 minutes*

Overview

Students consider whether various functions that we use when coding are associative.

Launch

In math, the Associative Property tells us that when adding three numbers or multiplying three numbers, it does not matter whether you start with the first pair or the last. In other words, we can change the groupings and get the same result!

- Can you predict which functions in Pyret are associative and which ones are not?
- Multiplication and addition are both commutative *and* associative. Do you think Pyret functions that are associative are *also* commutative?

Pose the above open-ended questions and invite students to discuss with a partner. Invite some students to share their thinking with the class.

Investigate

- Turn to Associativity and Code and open the Commutativity and Associativity Starter File.
- With your partner, make a prediction about whether each function is associative.
- Complete any Circles of Evaluation and code, then test the code to determine if the images produced are identical or not.

As students work, encourage them to *always* make predictions before testing the code. Similarly, the activity will be more valuable if students discuss *why* the code did or did not produce identical images. Debrief to ensure comprehension.

When everyone is finished, check in with students. Did everyone discover that *all* of the functions were associative? There is a good chance your students will wonder if *all* Pyret functions are associative!

- On Associativity and Code, we discovered that `overlay`, `beside`, `above` were all associative, but `blend-images` was not. Can you think of any other Pyret functions that are _not associative?

- *Answers may vary:* `string-contains` *is not associative. Other functions, such as* `triangle`, `rectangle`, *and others will produce errors if students attempt to apply the Associative Property.*

- Turn to Categorizing Functions, where you will synthesize what you have learned about the Commutative Property and Associative Property in both math and Pyret.
- Complete the table then respond to the questions.

Synthesize

- What did you learn about the Associative Property in Pyret? Did anything surprise you?
- How were the programming activities in this lesson similar to the paper-and-pencil activities? How were they different?